Hierarchy and Nestedness in Networked Organizations

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(Draft version–October 2017)

Abstract

This paper uses a network approach to study how a planner should design the interaction structure between a set of agents—who interacts with whom, and to what extent—to achieve her goals. The assumption is that the agents' actions exhibit strategic complementarities; such complementarities appear in the workplace and many social and economic settings. I show that the optimal interaction structure (as represented by the networks designed by the planner) is hierarchical such that all agents have different centrality, even if the agents are ex ante identical. Hierarchy and inequality thus arise endogenously. This is true even when the planner has a preference for equality. This paper is the first to characterize optimal networks with weighted and directed links under strategic complementarities. A new concept, nested core-periphery network, is proposed for the characterization.

JEL Classification: C72, D85.

Keywords: Hierarchy, core-periphery networks, nestedness, network games, strategic complementarity, optimal networks.

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1 Introduction

Hierarchy and nestedness are common features of many social, economic, and political organizations. In every typical firm, there is a CEO at the center. Then there is the board of directors, consisting of the CEO and other directors. Then there is the group of all managers. Think about the UK government. The prime minister is at the center; then it is the Cabinet, consisting of the prime minister and other most senior ministers; then all government ministers sit in Parliament, and are accountable to it. **Nestedness**[1] means that one of the core groups of members contains another—the group of all managers are the core members of the organization; but the board of directors constitutes a tighter core embedded in the group of managers; and within the board of directors, the CEO sits at the center.[2] This paper shows that hierarchy and nestedness are robust features of an optimal organization, even if all agents are ex ante identical and regardless of whether the planner seeks to maximize aggregate efficiency, or she is willing to make trade-offs between efficiency and equality.

Specifically, this paper uses a network approach to study how a planner designs the optimal **interaction structure**—who interacts with whom, and to what extent—between a set of agents. The agents play a game with **local complementarities**. Complementarity means that a best-response action of each agent is (weakly) increasing in the actions of others. The complementarities are local in the sense that not every agent influences every other agent to the same extent. An agent may impose greater influence on some agents than on others. The flow of complementarities is controlled by the interaction structure designed by the planner. For example, a school teacher (the planner) wants to improve the performance of students (the agents). The teacher can influence the interactions between students by assigning them to different groups, or nominate some students to be representatives. The complementarity is the peer effects between the students (Calvó-Armengol et al., 2009; Falk and Ichino, 2006). I model the interaction structure as a network—the agents are the nodes

[1] This paper uses **bold** to indicate defining a concept, and *italic* to highlight.

[2] See König et al. (2014) for more examples of nestedness in real-world organizations, including the Fedwire bank network (Soramäki et al., 2007), global arms trade network (Akerman and Seim, 2014), and the world trade network (De Benedictis and Tajoli, 2011).

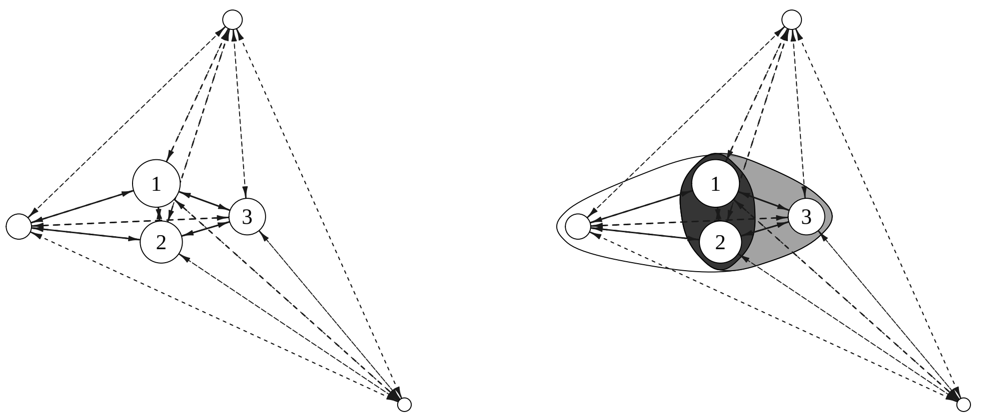


Figure 1: A nested core-periphery network. Dashed lines indicate weaker connections. The graph on the right highlights a nested sequence of cores: $\{1\} \subset \{1,2\} \subset \{1,2,3\}$.

within the network, and the bilateral influences between agents are the weighted and directed links that connect the nodes. The problem of finding an optimal interaction structure is equivalent to finding an optimal network among all networks with arbitrarily weighted and directed links.

My main result is that the optimal network is a *nested core-periphery network* with *strict hierarchy* (Proposition 2). Figure 1 shows a nested core-periphery network. In a **nested core-periphery network**, some agents are more central than others—both in terms of their **outward-centrality**, i.e., their outward influence on others, and their **inward-centrality**, i.e, the accumulation of others' influence on them. For example, agents 1, 2, 3 are more central than the rest in the network in Figure 1. Agents with greater centrality connect strongly to each other to form a **core**, while the less central ones connect only weakly to others. Depending on the threshold of centrality we set, we can construct *ex post* smaller cores or larger ones. The agents contained in a smaller core are all contained in larger cores. Hence, the different cores are nested. **Strict hierarchy** means that, in the optimal network, all agents have different centrality and, thereby, obtain unequal payoffs. Such hierarchy arises even if all agents are ex ante identical and regardless of whether the planner seeks to maximize aggregate performance (the sum of the efforts of the agents), or aggregate efficiency (the sum of payoffs of the agents), or she is willing to make trade-offs between aggregate efficiency and equality.[3]

Nested core-periphery structure of the optimal network arises for the following reason. Strengthening a link to an agent enhances complementarity and thus makes the agent exert more effort. Since strengthening a link is costly, the planner finds it optimal to first strengthen the links pointing to an agent with higher outward-centrality. The reason is that an agent with higher outward-centrality imposes greater influence on others. If the effort of an agent with higher outward-centrality increases, it spreads to others and leads to a greater improvement of aggregate performance compared to an agent with lower outward-centrality. But then, by strengthening the links pointing to agents with higher outward-centrality, the planner also increases their inward-centrality. As a result, if an agent is more central, she must be so both inwards and outwards. In the optimal network, all central agents are strongly connected to each other, whereas the less central ones are connected weakly to each other.

It is striking that strict hierarchy arises at optimum even if the agents are homogeneous and the planner is concerned with efficiency and equality at the same time. This result provides a foundation to the pervasiveness of hierarchy in the real world. The key insight is that complementarity generates a fundamental demand for creating a core group of agents and first enhancing the interactions among them. The core group of agents magnify each other's efforts through their strong, tight-knit connections. Their enhanced efforts then spill over to the network periphery and boost their efforts too. Hence, unless the only objective of the planner is to produce an egalitarian outcome, it never pays off to equalize the performance of any two agents. Within the core group, the planner continues to differentiate agents and create a tighter core group, etc. As a result, a nested, hierarchical sequence of core groups of agents emerges.

[3]The planner's optimization problem involves choosing policies to maximize an objective function defined on the actions of agents. The case where the planner seeks to maximize aggregate efficiency is captured by an objective function that is convex in the action of each agent. In contrast, an objective function that is strictly concave in the action of each agent captures the case where the planner is willing to sacrifice aggregate efficiency to reduce payoff difference between agents. My analysis covers both cases.

Alongside the interaction network, I also consider the tool of the planner such that she can make investment in agents to directly improve their performance. For example, a teacher decides her time to spend on supervising each student; a manager decides how to distribute a limited training budget between employees; and the government decides how to allocate funding among different education and research institutes. I show that, in the case where the interaction network is not optimized, the agents obtaining more investment from the planner need not be the ones exerting more effort. Also, high-ability agents need not form clusters, or receive more investment from the planner. In sharp contrast, when the interaction network is optimized, high-ability agents are clustered together at the very center of the network. Moreover, they are strongly connected to each other, and they receive more investment from the planner. This implies that, ceteris paribus, implementing the optimal network leads to more polarized payoffs between high-ability agents and low-ability ones.

This paper complements previous studies and makes two key contributions to the literature. *First*, this paper establishes a fundamental link between strategic complementarities and hierarchy. This result contributes to the large literature on organization design. The earlier literature examines optimal levels and the span of a firm to overcome incentive problems under asymmetric information, assuming from the outset that the firm takes a hierarchical form (Calvo and Wellisz, 1978, 1979; Qian, 1994).[4] Another strand of literature (Radner, 1993; Bolton and Dewatripont, 1994; Garicano, 2000; Van Zandt, 1999) sets aside incentive problems and investigates the efficient communication or knowledge acquisition networks. Like the latter ones, I set aside incentive problems. Due to complexity of the problem, most previous studies on organization design assume specific organization forms from the outset. Using a parsimonious model of strategic complementarities and the network approach, this paper characterizes the optimal organization structure without imposing prior restrictions on the form of the organization. The most closely related papers to this one are Belhaj et al. (2016) and Hiller (2017) . The two papers also investigate optimal networks in games with complementarities. However, they only examine efficient networks that maximize the sum of payoffs of agents. What if the planner is explicitly concerned with avoiding inequality

[4]More precisely, in network analysis terminology, Calvo and Wellisz (1978, 1979) and Qian (1994) assume from the outset that the organization takes the form of directed trees.

between agents? For example, a school teacher not only cares about the aggregate performance of students, but is particularly concerned with the welfare of those who might fail. In this case, it is not at all obvious from the outset that a hierarchical structure is optimal; I show that it is. This result is important in understanding hierarchy and inequality in nonprofit organizations such as the ones in the education sector.

Second, this paper makes a methodological contribution to the network literature. To my knowledge, this paper is the first to derive optimal networks among networks with weighted and directed links, taking strategic interactions between agents into account. Most previous analyses focus on simple networks—networks with links **unweighted**, i.e., each link takes the weight of either one or zero, and **undirected**, i.e., if agent A influences agent B, B must also influence A to exactly the same extent. However, in many social and economic settings, the complementarity technologies are not "simple". Complementarity may depend on a variety of factors including the time agents spend interacting, the quality of communication between agents, and the physical distance between them. Thus, in many relevant applications, complementarity should be treated as continuous. Moreover, complementarity may depend on formal administrative relationships or relative social status and thus be directed, i.e., one agent may have more influence on the other than vice-versa. My analysis captures these possibilities by examining networks with arbitrarily weighted and directed links. A new concept—nested core-periphery networks—is proposed to capture distinct features of the optimal network with weighted and directed links. The new concept extends previous notions—core-periphery networks[5] and nested split graphs[6]—defined on simple networks to networks with weighted and directed links. When restricting to simple networks, Belhaj et al. (2016) and Hiller (2017) find that a *non-hierarchical* network can arise at optimum such that every agent has the same centrality and obtains the same payoffs. I show that, if the planner can arbitrarily adjust the strength of each link, the optimal network must be hierarchical.

[5] A core-periphery network is a simple network in which the nodes are divided into two sets, the core set and the periphery set, such that two nodes are connected by a link if and only if one or both of them are in the core set.

[6] A nested split graph is a simple network in which the neighborhoods of nodes are nested such that, if node i has more neighbors than j, then all neighbors of j are neighbors of i.

The paper is organized as follows. Section 2 reviews the broader network literature and discusses in more detail the differences between this paper and previous analyses. Section 3 introduces the game of complementarities and the planner's problem. It also introduces the concepts of inward- and outward- centralities. Section 4 takes the network as exogenously given and derives equilibrium efforts of agents. It examines the planner's optimal investment in agents to directly affect their marginal return to efforts. Section 5 presents the main result of this paper; it introduces the notion of nested core-periphery network and characterizes the optimal network. Section 6 is the discussion and extension section. I discuss two topics in this section: one concerning the connections and distinctions between nested core-periphery networks and nested split graphs, and the other characterizing the equilibrium networks when the agents form their own links, instead of the planner designing them. Section 7 concludes. All proofs are collected in the Appendix.

2 Literature

I provide a brief review of the related literature in this section. I classify related papers into four categories (see Table 1): *i*) strategic interactions in exogenous networks, *ii*) formation of networks with weighted and directed links, *iii*) models that combine network formation and strategic interactions, and *iv*) designing policies to influence agents in the network, i.e., influencing the influencers. To the best of my knowledge, there is no existing paper that investigates the formation of general networks with weighted and directed links *and* takes strategic interactions between agents into account. Hence, existing literature is not ready to address the policy question—how to design an optimal network with weighted and directed links to affect the activities of agents.

2.1 Interactions in exogenous networks

Ballester et al. (2006) is the seminal paper characterizing equilibrium of network games. In their model, agents play a game in an exogenous network and they have linear-quadratic utility functions. The paper shows that interior equilibrium actions correspond to Bonacich

Table 1: Related literature

	Simple network (with unweighted and undirected links)	**General network** (with weighted links)
Interactions in exogenous networks	Ballester et al. (2006), Bramoullé and Kranton (2007), Bramoullé et al. (2014), Belhaj et al. (2014)	
Formation of networks	Bala and Goyal (2000), Jackson and Wolinsky (1996), etc.	Rogers (2006), Bloch and Dutta (2009), Ortega and Llovera (2016)
Interactions + Network formation	**Strategic complements**: Goyal and Moraga-Gonzalez (2001), Goyal and Vega-Redondo (2005), König et al. (2014), Baetz (2015), Hiller (2015), Belhaj et al. (2016), Jackson (2016) **Strategic substitutes:** Galeotti and Goyal (2010), Jackson (2016)	
Influencing the influencers	**Incomplete information**: Galeotti and Goyal (2009), Fainmesser and Galeotti (2015) **Complete information:** Bloch and Quérou (2013)	

centrality. Bramoullé and Kranton (2007) examine public good provision in a network. The game exhibits strategic substitutes between public good provision of neighbors, and they assume linear best-response functions. They find specialization in public good provision in equilibrium, i.e., some agents provide a lot while others free riding. Bramoullé et al. (2014) provide a general characterization of equilibria of network games, covering strategic substitutes as well as complements, interior equilibria as well as equilibria with inactive agents. Belhaj et al. (2014) examine strategic complements. The best-response function can be concave, or convex, relaxing the assumption of linear best-responses that previous analyses often assume. They identify conditions for the uniqueness of equilibrium. An interesting result is that, when available actions are bounded above, actions between agents may become *less* interdependent in a *denser* network, i.e, a shock on the action of a particular agent may be less likely to propagate throughout the network in a network with higher average degree.

2.2 Formation of networks with weighted and directed links

Rogers (2006) and Bloch and Dutta (2009) consider models in which each agent invests resources to form *weighted* links with others (but no other individual actions to take). Rogers (2006) shows that the equilibrium network and the efficient network coincide. Bloch and Dutta (2009) show that, if the return to a link is convex (no formal justification is given), both efficient and pairwise-stable networks are stars (with a single center agent). Ortega et al. (2016) examine a model in which planner invests to build weighted links given a set of agents. They also examine the case where agents form their own weighted links. In all three papers, no actions need to be taken by the agents. Hence, their models are silent about how the network affects the actions of agents and how the actions of agents feedback to determine the value of a link. Also, they only consider efficient networks that maximize the sum of the utilities of the agents. They do not examine the implications of a welfare function that takes the distribution of payoffs of the agents into account. This is critical because the novel and most surprising bit of my results is that, even if the planner is willing to make arbitrary trade-offs between efficiency and fairness, strict hierarchy arises.

Cabrales et al. (2011) consider a model in which both the strength of links (socialization) and production activities of the agents are endogenous. Each agent chooses a socialization level and a production level. The strength of a link between a pair of agents depends on the product of socialization levels chosen by the two agents. The model produces results regarding the equilibrium and efficient levels of socialization and production for each agent, but provides no results regarding network structure.

2.3 Interactions in endogenously formed simple networks

Goyal and Moraga-Gonzalez (2001) examine stable R&D collaboration network between horizontally related firms. Each firm chooses a cost-reducing action and their production level. The best-response cost-reducing action is linear in the actions taken by collaborating firms. This model is an extension of Goyal and Joshi (2003). In Goyal and Joshi (2003), firms do not choose cost reducing effort; instead, the mapping from collaboration network to cost is exogenously given.

Goyal and Vega-Redondo (2005) examine a setting in which each agent decides the linking with others as well as the action to coordinate on. The stage game is a 2-by-2 coordination game and the authors examine the stochastically stable network and action. The interesting finding is that whether the network is formed endogenously is critical to which action—the risk-dominant one or the efficient action—is more (stochastically) stable. If the network is exogenous, standard results (Kandori et al., 1993; Young, 1993) apply: the risk-dominant action is more stable. If the network is endogenous, which action is stochastically stable depends on the cost of forming a link. If the cost of forming a link is high, the efficient action is more stable.

Galeotti and Goyal (2010) combine Bramoullé and Kranton (2007), i.e., network games with strategic substitutes, and the network formation model of Bala and Goyal (2000), to explain "the law of the few", i.e., equilibrium networks are core-periphery networks with a small subset of core agents who exert much more effort to acquire information than others. Baetz (2015) investigates equilibrium networks (the agents form their own links) in games with strategic complements. Agents have utility functions concave in actions, and their best-response functions are also concave. Baetz shows that the network that arises in a strict equilibrium is a *multipartite graph*. A multipartite graph is generally *not* a nested core-periphery network or a nested split graph, and vice versa, except for some special cases such as empty networks, complete networks, or star networks.

König et al. (2014) present a dynamic model of network formation given the agents play games with strategic complements in the resulting network. They show that stochastically stable simple networks are nested split graphs. Belhaj et al. (2016) investigate efficient networks in games with strategic complements. The planner either maximizes the sun of the utilities of the agents, or the sun of the actions of the agents—the objective function of the planner is (weakly) convex in the actions of agents. Hiller (2017) investigates the equilibrium networks (the agents form their own links) in games with complements, and also examines the efficient network. Both Belhaj et al. (2016) and Hiller (2015) find that the efficient network is a nested split graph. These two papers are the most close ones to the current paper. However, *first*, Belhaj et al. (2016) and Hiller (2017), as well as König et al. (2014), examine only simple networks with unweighted and undirected links. *Second*, they only characterize efficient networks. In contrast, this paper covers the case in which planner seeks to maximize aggregate efficiency as well as the case in which the planner trade-offs efficiency and equality. *Third*, depending the cost of building a link, or the number of links allowed to build, a non-hierarchical complete network may arise at optimum in their models. I show that, with mild assumptions (to make sure that there are no isolated agents), the optimal network must exhibit strict hierarchy such that no two agents can have the same centrality.

Jackson (2016) analyzes the friendship paradox and the biases in the perceptions of social norms. Each agent knows only the degree distribution of the network, i.e., the *incomplete information approach* (Galeotti and Goyal, 2010). Two stages: first, each agent chooses his degree; and then an action level that complements the actions of his neighbors. This setup results in systematic biases in the perceptions of average level of actions.

2.4 Influencing the influencers in exogenous networks

Galeotti and Goyal (2009) and Fainmesser and Galeotti (2015) study pricing and seeding (i.e., offering sample products to some consumers) policies of firms to consumers in the present of network effects. They also study the value of collecting the network information of the consumers. In their model, each consumers (agent) only knows the *degree distribution* of population but not the identities of neighbors—the incomplete information approach. Their interest is on how aggregate network statistics such as average degree and dispersion of the degree distribution affect average level of seeding investment of firms. The impacts of micro-network statistics such as centrality of an agent is not explicitly explored.

Differently, Bloch and Quérou (2013) examine pricing of a monopoly in a model in which each consumer knows the identity of their neighbors. They then link the discriminatory price designed for each consumer to the consumer's Bonacich centrality. They examine undirected networks as well as directed ones, extending the analysis of Ballester et al. (2006).

3 The model

I introduce the model in this section. First, I describe the network game with strategic complementarities. Then I introduce the planner's objective function and constraints. Finally, I introduce the concepts of inward-centrality and outward-centrality, which are critical to subsequent analyses.

3.1 The network game with strategic complementarities

Let $N = \{1, 2, \ldots, n\}$ be a set of agents. The agents play a game with local complementarities. The effort of i is denoted by $x_i \geq 0$. The return to the effort of i has three determinants: a) the ability of i, b) the investment of the planner in the agent, and c) the effort of other agents. Let $\alpha_i > 0$ denote i's ability. Let $t_i \geq 0$ denote the planner's investment in agent i. To fix idea, think of the agents as school students, and the planner as a teacher. The teacher wants to improve students' effort. The teacher has two tools. One is dedicating more time to some students to improve their marginal return to effort. This is the investment t_i made by the planner.

Another tool is grouping the students in different ways with the aim to affect the **interaction structure** between them—who interacts with whom, and to what extent. Let $g_{ij} \geq 0$ be the strength of the directed influence from j to i. Notice that g_{ij} can take any non-negative value, and g_{ij} need not equal to g_{ji}. Let $G = (g_{ij}) \in \mathbb{R}_+^{n \times n}$ be the matrix that specifies the strength of the directed influences, with diagonal element $g_{ii} = 0$ for each $i \in N$. The assumption is that, if g_{ij} increases, the improvement in j's effort leads to a greater increase in i's effort. Think of the teacher-students example. Calvó-Armengol et al. (2009) show that peer effects exist among school students in terms of exerting effort. Their structural model builds on exactly the same linear-quadratic utility function introduced below. They show that this utility function fits well with the data.

More precisely, each agent simultaneously chooses an effort $x_i \geq 0$ to maximize

$$u_i(x_i, x_{-i}; t_i, G).$$

Assumption 1. Let $\lambda > 0$ be a scalar measuring the average level of complementarity. We have

$$u_i(x_i, x_{-i}; t_i, G) = \left[\alpha_i + v(t_i) + \lambda \sum_{j \in N} g_{ij} x_j\right] x_i - \frac{1}{2} x_i^2, \tag{1}$$

where $v' > 0, v'' < 0$ and $v'(0) = \infty$.

The utility function has two terms. The first term is the return to effort. The influence from j to i, g_{ij}, determines the extent to which the marginal return to the effort of i is increasing in the effort of j. The second term is the cost of exerting effort, which is increasing in x_i quadratically. Due to its intuitive interpretation and tractability, this linear-quadratic utility function is widely used in the literature of social interactions (e.g., Goyal and Moraga-Gonzalez 2001; König et al. 2014; Belhaj et al. 2016; Akerlof 1997; Glaeser and Scheinkman 2001; Calvó-Armengol et al. 2009).

Given the utility function, the best-response effort of i is linearly increasing in the effort of others:

$$x_i(x_{-i}) = \alpha_i + v(t_i) + \lambda \sum_{j \in N} g_{ij} x_j. \tag{2}$$

Note that the ability of the agent, α_i, and the investment of the planner, t_i, are linearly separable in the best-response function. I could instead allow for an interaction effect between α_i and t_i; the qualitative results would not change. But assuming separability highlights the unique features of the model. I aim to show that, even if α_i and t_i are linearly separable, a positive correlation between them emerges when interaction structure is optimized.

3.2 The planner's objective

The planner cares about the effort exerted by the agents. The investment $\boldsymbol{t} = (t_1, \ldots, t_n)$ and the interaction structure G are two policy tools of the planner. More precisely, the planner chooses

1. $t_i \geq 0$ for each $i \in N$, and
2. g_{ij} as well as g_{ji} for each $i, j \in N$ with $i \neq j$

to maximize the **objective function** $W(x_1, \ldots, x_n)$.

Assumption 2.

$$W(x_1, \ldots, x_n) = \sum_{i \in N} f(x_i),$$

where f is twice differentiable and $f' > 0$.

Let me emphasize the following: I have not imposed any restriction on the concavity of the objective function—f, and hence W, can be *linear* or *strictly convex* or *strictly concave*. A strictly concave objective function captures the cases where the planner considers both efficiency and fairness.

The following examples show the variety of cases covered by the setup.

Example 1 (Utilitarian planner)**.**

$$W(x_1, \ldots, x_n) = \sum_{i \in N} u_i(x_i, x_{-i}).$$

To see that our setup covers this case, substituting the best-response effort (2) to utility function (1), we obtain

$$u_i(x_i, x_{-i}) = \frac{1}{2} x_i^2.$$

Example 2 (Trade-off between efficiency and equality)**.**

$$W(x_1, \ldots, x_n) = \sum_{i \in N} \ln x_i.$$

In this case, the planner has greater concerns for those agents who exert less effort. Hence, the planner is willing to sacrifice some level of average performance to reduce the differences between agents, i.e., the planner is willing to make trade-offs between efficiency and equality. In addition, maximizing $\sum \ln x_i$ is equivalent to maximizing $\prod x_i$. Hence, this example also shows that separability of efforts x_i across i is not a necessary condition for subsequent results.

3.3 The planner's constraints

The planner faces three constraints: *a*) limited investment resources, *b*) bounded intensity of interactions, and *c*) utility maximization of each agent.

First, the planner has limited resources to invest. Let $\bar{t} > 0$ be the total resources that the planner possesses. The total investment cannot exceed $\bar{t}$:

$$\sum_{i \in N} t_i \leq \bar{t} \quad \text{(limited investment resources).} \tag{3}$$

Second, the total intensity of interactions is bounded. If the planner strengthens the influences between some agents, she gives up the opportunity to strengthen the influences between other agents. Let $\bar{\phi} \in \mathbb{R}_{++}$ be an upper bound of the total intensity of interactions that can be achieved. We have

$$\sum_{i,j \in N, i \neq j} \phi(g_{ij}) \leq \bar{\phi} \quad \text{(bounded intensity of interactions).} \tag{4}$$

I make the following assumption about $\phi(.)$. It is strictly convex, so that it gets harder and harder to strengthen the influence between any pair of agents. Also, $\phi'(0) = 0$, so that at optimum all links take positive values (they can still be negligibly small).[7]

Assumption 3. *ϕ is increasing and strictly convex, i.e., $\phi'(g_{ij}) > 0$ for $g_{ij} > 0$, and $\phi'' > 0$. Also, $\phi'(0) = 0$.*

Third, after the planner implements $\boldsymbol{t} = (t_1, \ldots, t_n)$ and G, the agents choose their effort simultaneously to maximize their utility. That is,

$$x_i \in \arg\max_{x_i \geq 0} u_i(x_i, x_{-i}; t_i, G). \tag{5}$$

I summarize the planner's problem below.

[7] This assumption highlights unique features of the model. I shall argue that solving the planner's problem leads to a hierarchical organization such that the agents can be strictly ranked based on their centrality (a concept I define in the next subsection), even if all agents are ex ante identical. If $\phi'(0)$ instead takes a high value, then some agents may result in the same centrality simply because all links leading to them take the corner solution value of zero.

Definition (The planner's problem)**.**

$$\max_{\boldsymbol{t}\geq 0, G\geq 0} \quad W(x_1,\ldots,x_n) = \sum_{i\in N} f(x_i)$$

$$\text{s.t.} \quad (1) \quad \sum_{i\in N} t_i \leq \bar{t}$$

$$(2) \quad \sum_{i,j\in N, i\neq j} \phi(g_{ij}) \leq \bar{\phi}$$

$$(3) \quad x_i \in \arg\max_{x_i} u_i(x_i, x_{-i}; \boldsymbol{t}, G) \quad \forall i \in N.$$

3.4 Network and centrality

Before the analysis, I introduce the concepts of inward-centrality and outward-centrality.

A **network** consists of *a*) a set of nodes and *b*) the weighted and directed links that connect the nodes. Each network can be represented by a unique non-negative square **adjacency matrix** $G = (g_{ij}) \in \mathbb{R}_+^{n\times n}$, with the dimension of the matrix n equal to the number of nodes in the network, and g_{ij} indicating the link pointed from node j to node i. If g_{ij} takes a higher value, the link from j to i is said to be stronger. With abuse of notation, I use G to denote both the network and the corresponding adjacency matrix.

In network G, a **walk** from j to i is a list of nodes $(j, k_1, k_2, \ldots, k_q, i)$ such that each of the links in the list $(g_{k_1 j}, g_{k_2 k_1}, \cdots, g_{i k_q})$ is strictly positive. Let G^k denote the kth power of G. Let $g_{ij}^{[k]}$ denote the ijth element of G^k, which counts the number of all walks of length k leading from j to i in the network.[8] Let m_{ij} be the total number of walks of all lengths leading from j to i, with each walk discounted accordingly to the length of the walk. That is,

$$m_{ij} = \sum_{k=0}^{\infty} \lambda^k g_{ij}^{[k]}, \tag{6}$$

where $\lambda > 0$ is the discounting factor. With this definition, m_{ij} has an intuitive interpretation: it summarizes direct as well as indirect influences from j to i through all possible walks in the network, with the influence through each walk discounted accordingly to the length of the walk.

[8] Note that $g_{ij}^{[k]}$ is *not* the kth power of g_{ij}.

Although it is intuitive to define m_{ij} by the above expansion, it is more convenient to work with the following equivalent formula:

$$M = [I - \lambda G]^{-1},$$

whose ijth element is m_{ij}. To make sure that the series (6) converges and $[I - \lambda G]$ is invertible so that m_{ij} is well defined, I assume the following sufficient and necessary condition throughout the paper:

Assumption 4. *Let $\mu(G)$ be the largest real eigenvalue of G. We have $\lambda < 1/\mu(G)$ for each feasible G under constraint (4).*

The assumption requires that the average level of complementarity, λ, is not too large. If λ is too large, each agent would exert infinitely high level of effort without bound, and thus an equilibrium for the complementarity game does not exist.[9]

The following concepts adopt Bonacich centrality (Bonacich, 1987; Bonacich and Lloyd, 2001) to networks with directed links.

Definition (Centrality)**.** Consider network G defined on set of nodes N. Let $z_i \in \mathbb{R}$ be some attribute of $i \in N$ that is determined exogenously to the network. The **inward-centrality** of i with **base value** $\boldsymbol{z} = (z_1, \ldots, z_n)$ is

$$\mathcal{I}_i(z; G) = \sum_{j \in N} m_{ij} z_j;$$

The **outward-centrality** of i with base value z is

$$\mathcal{O}_i(z; G) = \sum_{j \in N} m_{ji} z_j.$$

Intuitively, inward-centrality $\mathcal{I}_i(z; G)$ summarizes the influences that a node *receives*

[9]For a proof of that $\lambda < 1/\mu(G)$ implies the invertibility of $[I - \lambda G]$, see Debreu and Herstein (1953). Note also that, $\mu(.)$, the mapping from square matrix to its largest eigenvalue is continuous (Allan and Dales, 2011, p. 168, Proposition 4.21). Hence, if G is such that $\mu(G) < 1/\lambda$, then there is $\delta > 0$ such that, for each G^ϵ with $||G^\epsilon - G|| < \delta$, we have $\mu(G^\epsilon) < 1/\lambda$ too. The norm $||.||$ here can be defined by Euclidean distance. In other words, if G satisfies $\mu(G) < 1/\lambda$, and G^ϵ is a small enough deviation from G, then G^ϵ satisfies $\mu(G^\epsilon) < 1/\lambda$ too.

from all other nodes in the network, whereas outward-centrality $\mathcal{O}_i(z;G)$ summarizes the influences that a node *imposes on* all other nodes.

4 Exogenous interaction structure

In this section, I analyze the benchmark case where the planner takes the interaction structure as exogenously given. I show that some properties that seem natural may not hold in an exogenously given network. However, these properties do hold in the optimal interaction structure I characterize in the next section. This comparison highlights the unique features of the optimal interaction structure. Much of the analysis in this section builds on Ballester et al. (2006).

To start with, let $\nabla W(\boldsymbol{x}) \in \mathbb{R}^n$ denote the gradient of $W(\boldsymbol{x})$, i.e.,

$$\nabla W(\boldsymbol{x}) = \left(f'(x_1), f'(x_2), \ldots, f'(x_n)\right).$$

With abuse of notation, let $v(\boldsymbol{t}) \in \mathbb{R}^n$ denote the vector whose ith element is $v(t_i)$. The following proposition characterizes the investment decision of the planner and the equilibrium effort of the agents.

Proposition 1 (Investment in agents)**.** *Consider the planner's partial problem such that the planner takes G as exogenously given.*

1. *A solution $\boldsymbol{t} = (t_1, \ldots, t_n)$ exists such that, for each $i, j \in N$, $t_i \geq t_j$ if and only if*

$$\mathcal{O}_i\left(\nabla W(\boldsymbol{x}); G\right) \geq \mathcal{O}_j\left(\nabla W(\boldsymbol{x}); G\right).$$

2. *The equilibrium effort of $i \in N$ is*

$$x_i = \mathcal{I}_i\left(\boldsymbol{\alpha} + v(\boldsymbol{t}); G\right) = \sum_{j \in N} m_{ij} \Big[\alpha_j + v(t_j)\Big].$$

Proof. All proofs are provided in the Appendix. □

The proposition is illustrated as follows. First, the planner invests more in those with higher outward-centrality $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G)$. This occurs because agents with greater outward-centrality are more influential, so that they have greater potential to improve the effort of others. Hence the planner finds it optimal to invest more resources in them. In an organization, we can think of positions with greater outward-centrality as those higher up in the hierarchy[10], or at the center of the organization's communication network, or occupying a more visible working spot in the corporation building (Brass, 1984, 2003; Pfeffer, 1992).

Second, the equilibrium effort of agent i equals precisely to his inward-centrality in the network, with base value $\alpha + v(t)$. This can be further expressed by a weighted sum of $\alpha_j + v(t_j)$ over the agents, and the weight put on the attributes of j is m_{ij}. This is intuitive because m_{ij} summarizes all possible channels that j influences i.

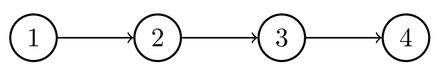


Figure 2: Directed Line

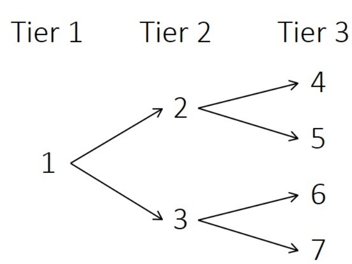


Figure 3: Directed tree

I make the following remarks to highlight the unique features of the optimal network characterized in the next section.

Remark 1. When the network is exogenously given to the planner, the agents who receive more investment need not be the ones who exert more effort.

[10]In Calvo and Wellisz (1979)'s hierarchy organization model, the production workers or supervisors at every level are supervised by the supervisors at the higher level. If a supervisor shirks, the supervisors under him also shirk, which leads to further shirking of the supervisors and production workers at lower levels.

In an arbitrary network, inward-centrality and outward-centrality may differ remarkably. Hence, the agents exerting more effort can be different from the ones receiving more investment from the planner. For example, consider the directed line in Figure 2, and the directed tree in Figure 3. As indicated by the arrows, the influences only go in one direction. In the directed line, agent 1 influences agent 2 who influences agent 3, etc. In this example, agent 1 has highest outward-centrality and thus receive highest investment from the planner. However, agent 4 has highest inward-centrality and thus exert the most effort. Likewise, in the directed tree, the agents in tier 1 and tier 2 have higher outward-centrality and thus receive higher investment from the planner, but it is the agents in tie 3 who exert the most effort.

Remark 2. Later I show that, when G is optimized, the high-ability agents are clustered to form a "core"; however, this is *not* due to the complementarity between the ability of different agents.

To see this, substituting each agent's equilibrium effort into the planner's objective function, we obtain

$$\frac{\partial^2 W}{\partial \alpha_i \partial \alpha_j} = \sum_{k \in N} f''(x_k) m_{ki} m_{kj}.$$

Observe from the above expression that, when $f'' < 0$, the ability of different agents are substitutes. Hence, suppose that the network is exogenously given, but the planner can decide the positions of the agents in the network. Then the planner may even find it optimal to separate the high-ability agents instead of clustering them. As we will see, this contrasts sharply with what happens when the planner can freely shape the network.

5 Optimal interaction structure

I analyze the optimal interaction structure in this section. The main result is that the optimal network is a nested core-periphery network—a generalization of core-periphery network to general networks with weighted and directed links. Also, the optimal nested core-periphery network must be a non-trivial one such that no two agents can have the same centrality, i.e., hierarchy necessarily arises.

5.1 The optimal network

The optimal network is derived from the following observations.

Lemma 1. *Let (G, t) solve the planner's problem. Let $x_i = \mathcal{I}_i(\alpha + v(t); G)$ be the equilibrium effort. Consider agents i and j.*

1. *If and only if $\mathcal{I}_i(\alpha + v(t); G) \geq \mathcal{I}_j(\alpha + v(t); G)$, then $g_{ki} \geq g_{kj}$ for each $k \neq i, j$.*

2. *If and only if $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G) \geq \mathcal{O}_j(\nabla W(\boldsymbol{x}); G)$, then $g_{ik} \geq g_{jk}$ for each $k \neq i, j$.*

3. *If and only if $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G) \geq \mathcal{O}_j(\nabla W(\boldsymbol{x}); G)$, then $\mathcal{I}_i(\alpha + v(t); G) \geq \mathcal{I}_j(\alpha + v(t); G)$.*

The first statement of the lemma compares links *starting from* an agent with higher *inward*-centrality with links starting from an agent with lower inward-centrality. If agent i has higher inward-centrality than agent j, then the link starting from i to a third agent k is stronger than the link starting from j to k, for each $k \neq i, j$. The reason for this is that, if i has higher inward-centrality, i exerts more effort. Hence the planner finds it optimal to strengthen the links starting from i to other agents in order to increase i's influences on other agents. The second statement of the lemma compares links *pointing to* an agent with higher *outward*-centrality with links pointing to an agent with lower outward-centrality. If agent i has higher outward-centrality than agent j, then the link pointing from a third agent k to i is stronger than the link pointing from k to j, for each $k \neq i, j$. This is optimal because strengthening the links pointing to i makes i to exert more effort. If i has high outward-centrality, this motivates others to exert more effort.

The first two observations imply the third—if an agent is more central in the optimal network, she must be so both inwards and outwards. This result is interesting because we ex ante allow for arbitrarily weighted and directed links. Hence, it might arise such that some agents have higher outward-centrality while the others have higher inward-centrality. However, in the optimal network, the asymmetry of complementarity flow in the two directions is bounded: the ordering of agents based on outward-centrality coincides with the ordering based on inward-centrality.

The above lemma implies a *core-periphery structure* of the optimal network. We can separate the agents into two groups—the **core group** contains agents with higher centrality, and the **periphery group** contains agents with lower centrality. The agents in the core group are strongly connected to each other (in both directions), whereas the agents in the periphery group are connected to each other weakly. In the literature, a core-periphery network is formally defined as follows (Goyal, 2007, p. 11). A network G is **simple** if $g_{ij} \in \{0, 1\}$ and $g_{ij} = g_{ji}$ for each $i, j \in N$. A network is **connected** if, for each two agents i and j in the network, there is a walk either from i to j or from j to i or both. A network G is a **core-periphery network** if G is simple, connected, and the set of agents N can be partitioned into two non-empty sets $N^C \subset N$ and $N^P \subset N$ such that

$$g_{ij} = \begin{cases} 1 & \text{if } i, j \in N^C \quad \text{(the core)}, \\ 0 & \text{if } i, j \in N^P \quad \text{(the periphery)}. \end{cases}$$

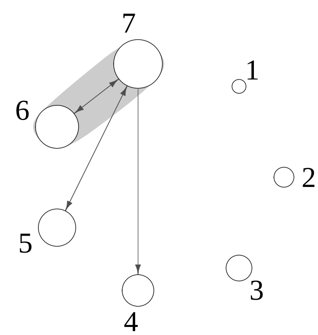
However, this standard notion of core-periphery network is inadequate to capture the characteristics of the optimal network with weighted and directed links for two reasons. First, the links between the periphery nodes need not take the exact value of zero, although they take lower values than links between the core nodes. Second, simply classifying the nodes into two sets may leave out some unique properties of the network. For example, in the optimal network in our context, depending on the threshold of centrality, we can construct a smaller core or a larger one. However, all nodes contained in a smaller core must be contained in a larger core. Hence, the constructed cores are nested. The standard notion of core-periphery network fails to capture this nestedness structure. This motivates the new notion—nested core-periphery networks (see Figure 4 for a graphical illustration).

Definition (Nested core-periphery networks)**.** Let G be a network defined on the set of agents N. G is a **nested core-periphery network** if the agents in N can be totally ordered such that, for each $i, j \in N$, one of the following must hold:

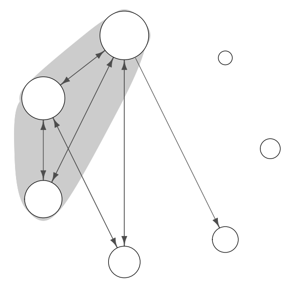
1. $g_{ik} \geq g_{jk}$ and $g_{ki} \geq g_{kj}$ for each $k \in N, k \neq i, j$;

2. $g_{ik} \leq g_{jk}$ and $g_{ki} \leq g_{kj}$ for each $k \in N, k \neq i, j$.

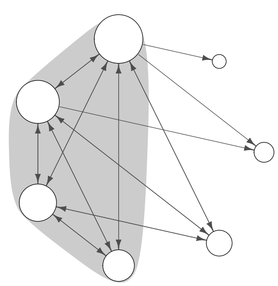
(a) Top 10% strongest links



(b) Top 25% strongest links



(c) Top 50% strongest links



(d) Top 75% strongest links

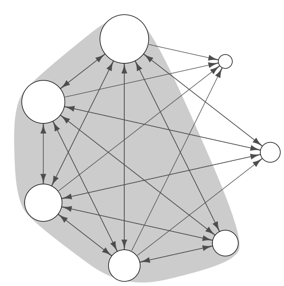


Figure 4: A nested core-periphery network. The size of each node shows the inward-centrality of the node. The four graphs show the same underlying network: (a) only shows the top ten percentile links according to the weight of the links, (b) shows the top twenty-five percentile links, (c) shows the top fifty percentile links, and (d) shows the top seventy-five percentile links. The gray area highlights the core nodes.

If G is a nested core-periphery network, there exists a function $c : N \to \mathbb{R}$ to represent the ordering of nodes such that, if $c(i) \geq c(j)$, we have $g_{ik} \geq g_{jk}$ and $g_{ki} \geq g_{kj}$ for each $k \neq i, j$. The function $c(.)$ is called a **coreness measure** of G,[11] and $c(i)$ is the **coreness** of i. The adjective "nested" has two meanings. One is that we can construct ex post a nested sequence of core groups of agents based on $c(.)$, as previously described. The other is that the neighborhoods of the agents are nested. More precisely, consider an arbitrary threshold $r \geq 0$. If $g_{ik} \geq r$, agent k is an **inward-neighbor** of agent i. Likewise, if $g_{ki} \geq r$, k is an **outward-neighbor** of i. If G is a nested core-periphery network, and i has higher coreness than j, then all inward-neighbors of j (excluding i) are inward-neighbors of i, and all outward-neighbors of j (excluding i) are also outward-neighbors of i.

Example (Nested core-periphery networks)**.** The following are special cases of nested core-periphery networks:

1. **empty networks** such that $g_{ij} = 0$ for each two agents i and j in the networks;

2. **simple complete networks** such that $g_{ij} = 1$ for each $i, j \in N$, $i \neq j$;

3. **weighted complete networks** such that there is a constant $\bar{g} > 0$ such that all links have weight $\bar{g}$, i.e., $g_{ij} = \bar{g}$ for each $i, j \in N$, $i \neq j$;

4. **nested split graphs**. In a simple network, a **neighbor** of i is an agent k with $g_{ik} = 1$. A network G is a nested split graph if G is simple, and the neighborhoods of the agents in G are nested such that, for each two agents i and j, if i has equal or more neighbors than j, then all neighbors of j (excluding i) are also neighbors of i. Belhaj et al. (2016) and Hiller (2017) show that, when restricting to simple networks, the efficient network—the one that maximizes the sum of the utilities of the agents—is a nested split graph. Note that empty networks and simple complete networks are special cases of nested split graphs.[12]

The above examples show that the class of nested core-periphery networks contains

[11] It is not unique.

[12] I further compare my characterization with the ones obtained by Belhaj et al. (2016) and Hiller (2017) in subsection 6.2.

empty networks and complete networks as special cases. In an empty network or a complete network, all agents have the same (inward- as well as outward-) centrality. In other words, these network are symmetric and non-hierarchical. A key question is then whether a hierarchical structure necessarily arises at optimum. The following proposition provides the main result of this paper—the optimal network is a nested core-periphery network with **strict hierarchy** such that no two agents can have the same centrality, even if all agents are ex ante identical. The optimal network must be a hierarchical one.

Proposition 2 (Optimal network)**.** *Let (G, t) solve the planner's problem, and $\boldsymbol{x}$ be the equilibrium effort under (G, t), i.e., $x_i = \mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G)$ for each i.*

1. *G is a nested core-periphery network, and the coreness of i is measured by*

$$\mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G).$$

2. *For each $i, j \in N, i \neq j$, we have $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G) \neq \mathcal{O}_i(\nabla W(\boldsymbol{x}); G)$ and $\mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G) \neq \mathcal{I}_j(\boldsymbol{\alpha} + v(\boldsymbol{t}); G)$. This holds even if $\alpha_i > 0$ is the same for each $i \in N$.*

Remark. The planner's investment $\boldsymbol{t}$ plays no role in the proof. The proposition holds regardless of whether investment $\boldsymbol{t}$ is optimized. (I discuss the planner's investment, together with positioning the agents, in the next subsection.)

The first statement says that the optimal network is a nested core-periphery network, and inward-centrality $\mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G)$ is used to measure the coreness of the agents. The second statement says that the agents must be strictly ranked such that no two agents can have the same outward-centrality or inward-centrality.

Combining Propositions 1 and 2, the following picture emerges. The planner finds it optimal to build a network exhibiting hierarchical structure such that all agents have different centrality. But once the planner does this, she also differentiates the investment between agents—the planner allocates more investment to agents with higher outward-centrality. In a nested core-periphery network, inward-centrality and outward-centrality induce the same ordering of agents. As a result, the agents who receive more investment also exert more

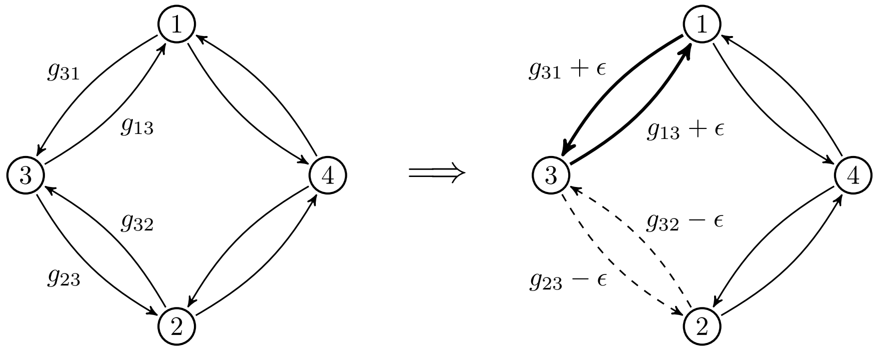


Figure 5: If agents 1 and 2 have the same coreness, an arbitrage opportunity exists so that the planner's objective function can be strictly improved.

effort. In terms of welfare, given the linear-quadratic utility of the agents, their utility in equilibrium is simply $u_i = \frac{1}{2}x_i^2$. Hence, agents with higher centrality obtain higher utility. Since no two agents can have the same centrality in the optimal network, no two agents obtain the same utility—inequality is inevitable. In short, even if all agents are ex ante identical, and even if the planner has an arbitrarily concave objective function, strict hierarchy and inequality emerge at optimum.

The first statement of Proposition 2 follows immediately from Lemma 1 and the definition of nested core-periphery networks. The second statement of the proposition builds on the following observation.[13] If two agents, i and j, have the same inward-centrality, there exists an arbitrage opportunity as follows. First, we reduce the weights of the links pointing to or starting from i—i.e., g_{ik} and g_{ki} for $k \neq i, j$—by a small amount. Meanwhile, we increase the weights of the links pointing to or starting from j—i.e., g_{jk} and g_{kj} for $k \neq i, j$—by the same amount. This simultaneous adjustment ensures that the constraint of the total intensity of interactions remains satisfied. However, the adjustment strictly increases the equilibrium efforts of almost all agents. The only agent who *might* reduce her effort is i. But even if i reduces her effort, this is fully compensated by the improvement in j's effort, regardless of the concavity of the planner's objective function. Hence, the adjustment leads to strict improvement of the planner's objective.

[13] The observation extends Belhaj et al. (2016)'s argument to the case where the planner can decide the direction and the strength of each link. In Belhaj et al. (2016) , the planner is constrained to choose links $g_{ij} \in \{0, 1\}$ and $g_{ji} = g_{ij}$.

To outline the proof in a greater detail, consider the example in Figure 5. Let the network on the left-hand side in the figure be a nested core-periphery network (there may be other links in the network but they are not shown), and all four agents have the same ability α_i. Suppose that agents 1 and 2 have the same inward-centrality $\mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G)$. I now show that the planner finds it strictly better to re-allocate an $\epsilon > 0$ amount of weight from links g_{23} and g_{32} to links g_{13} and g_{31}. I show this by considering how agents' efforts transit from the equilibrium under the initial network to the equilibrium under the new network. *Step 1*: To start with, let agents 3 and 4 exert the same amounts of effort as they do in the initial network. But 1 and 2 adjust their efforts according the new network on the right-hand side of the figure. Since the link from 3 to 1, g_{31}, is stronger than before, 1 increases his effort; meanwhile, 2 reduces her effort. However, *a*) given linear best-response functions, the increase in 1's effort is *equal* to the reduction of 2's effort in magnitude. Moreover, since 1 and 2 initially have the same inward-centrality, they initially exert the same amount of effort. Hence, *b*) for small ϵ, the planner cares about the efforts of 1 and 2 *equally* at the margin, regardless of the concavity of the planner's objective function. As a result, the planner finds the adjustment so far leading to an outcome at least as good as before, and this is not affected by the concavity of the objective function.

Step 2: After 1 and 2 have updated their efforts, let 3 and 4 also update their efforts. Since we increase the influence from 1 to 3 and 1 increases his effort, 3 will strictly increase his effort too. How about agent 4? The influences from 1 to 4 and from 2 to 4—i.e., g_{41} and g_{42}—are the same as before. But then, if the initial network is an optimal core-periphery network, and 1 and 2 initially have the same inward-centrality, we must have $g_{41} = g_{42}$. In other words, when updating his effort, 4 puts the same weight on 1 and 2 (just as the planner does!). Since the increase of 1's effort is at least as great as the reduction of 2's effort in magnitude, 4 will exert an effort at least as great as before. To summarize the changes so far: 1 and 3 strictly increase their efforts; 4 exerts an effort at least as great as before; 2 reduces her effort but this is fully compensated by the increase of 1's effort. Taking together, the planner finds the new outcome strictly better. In fact, given the complementarity technology, the agents will not settle down their efforts on the current status. They will adapt their efforts to each other and increase efforts further, finally converging to the equilibrium under the new network.[14]

The above argument shows that re-allocating some weight from links g_{23} and g_{32} to links g_{13} and g_{31} leads to a strictly better outcome. By the same argument, the planner finds it strictly better to re-allocate some weight from links g_{2k} and g_{k2} to links g_{k1} and g_{1k} for each $k \neq 1, 2$. Consequently, 1 ends up having higher outward-centrality and higher inward-centrality than 2, although ex ante there is nothing special about 1 compared to 2.

5.2 Positioning heterogeneous agents

A question remains unanswered: if the agents have different ability α_i, who should be placed to a position with higher centrality? The following proposition answers this question. The proposition says that agents with higher ability are placed to positions with higher outward-centrality.

Proposition 3 (Positioning agents). *Let $(\boldsymbol{t}, G)$ solve the planner's problem. If i and j are such that $\alpha_i > \alpha_j$, then*

$$\mathcal{O}_i(\nabla W(\boldsymbol{x}); G) > \mathcal{O}_j(\nabla W(\boldsymbol{x}); G)$$

and

$$\mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G) > \mathcal{I}_j(\boldsymbol{\alpha} + v(\boldsymbol{t}); G).$$

Such placement of agents in the optimal network has three implications. First, high-ability agents now receive more investment from the planner, even if investment t_i and ability α_i are linearly separable in the agents' best-response function.

Second, in the optimal network, the agents who have higher outward-centrality also have higher inward-centrality. Hence, they also exert more effort and thus get more utility. In contrast, in the remarks following Proposition 1, I show that these correlations need not hold in an exogenously given network.

[14] If the original network G satisfies $\mu(G) < 1/\lambda$ and thus a unique equilibrium effort profile exists, then a small enough deviation G^ϵ satisfies $\mu(G^\epsilon) < 1/\lambda$ too and thus a unique equilibrium effort profile also exists for G^ϵ. See footnote 9.

Third, because of the nested core-periphery structure of the optimal network, the high-ability agents are now clustered altogether to the very center of the network. The high-ability agents are strongly connected to each other, leading to greater benefits from their higher ability. As a result, other things equal, high-ability agents and low-ability ones obtain more polarized payoffs.

6 Discussion and extensions

I discuss two topics in this section. First, building on Bala and Goyal (2000), I consider a network formation game in which each agent chooses the weights of links related to them. I show that the resulting networks in equilibrium are again nested core-periphery networks. However, there are two distinctions between equilibrium networks and the efficient one: first, standard equilibrium notion cannot rule out non-hierarchical networks; second, all equilibrium networks are inefficient. Then, I discuss the connections and the distinctions between my characterization of optimal networks among networks with weighted and directed links and previous characterization restricted to simple networks.

6.1 Spontaneous networks

So far I have considered how a planner organizes the interactions to maximize the welfare of the agents. Now, I examine what if the agents choose their own connections. I show that the resulting network is also a nested core-periphery network. However, all links in the resulting network are weaker than the level that would be implemented by the planner. Hence, the resulting network is inefficient.

Consider the following two-stage **strategic network formation game**. To focus on network structure, let $\alpha_i = 1$ for each agent. In the first stage, each agent chooses a $(n-1)$-dimensional non-negative vector

$$\boldsymbol{g}_i = (g_{i1}, g_{i2}, \ldots, g_{i(i-1)}, g_{i(i+1)}, \ldots, g_{in}) \in \mathbb{R}_+^{n-1},$$

where g_{ij} indicates the direct influence from j to i. Let $\boldsymbol{g}_{-i}$ denote the collection of decisions other than i's. Let $G \in \mathbb{R}_{+}^{n\times n}$ denote the matrix that collects the decisions of all agents, with zeros on the main diagonal. With abuse of notation, I also use $(\boldsymbol{g}_i, \boldsymbol{g}_{-i})$ to denote the matrix G. In the second stage, each agent observes G, and exerts effort $x_i \geq 0$. The game ends and the payoffs for i are

$$\pi_i(\boldsymbol{x}, G) = \left[1 + \lambda \sum_{j\in N} g_{ij}x_j\right]x_i - \frac{1}{2}x_i^2 - \sum_{j\in N, j\neq i} \phi(g_{ij}).$$

The first two terms are the benefits and the costs of exerting effort x_i, respectively. The third term is the cost of establishing the complementarities, with $\phi' > 0, \phi'' > 0$.

For example, think of the agents as school students who a) seek out their study-buddies, $\boldsymbol{g}_i$, and b) decide how much effort to put to study, x_i. Study-buddies of i are those students who influence i's study positively. Each student not only need to decide who to be their study-buddies, but also how much time and effort and other resources to spend in order to benefit from each of his study-buddy. The marginal cost of establishing benefits from each study-buddy is increasing. Hence, $\boldsymbol{g}_i$ takes a continue value and $\phi(.)$ is convex.

A **subgame perfect equilibrium** $(\boldsymbol{x}, G)$ of the game is such that, for each $i \in N$,

1. $x_i = \mathcal{I}(1; \boldsymbol{g}_i, \boldsymbol{g}_{-i})$, and

2. $\boldsymbol{g}_i \in \arg\max_{\boldsymbol{g}_i} \pi_i(\mathcal{I}(1; \boldsymbol{g}_i, \boldsymbol{g}_{-i}), \boldsymbol{g}_i, \boldsymbol{g}_{-i})$.

Applying Lemma 2 in the Appendix, the second condition above leads to the following first-order condition for g_{ij} with $i \neq j$:

$$\lambda\left[\mathcal{I}_i(1; G)m_{ii}\right]\mathcal{I}_j(1; G) - \phi'(g_{ij}) = 0. \tag{7}$$

In fact, this first-order condition is both necessary and sufficient for a network G to be an equilibrium network.

However, an **efficient network** G^e is one that maximizes $\sum_{i\in N} \pi_i$. It satisfies, for each element g^e_{ij},

$$\lambda\Big[\mathcal{I}_i(1;G^e)m_{ii}+\sum_{k\in N,k\neq i}\mathcal{I}_k(1;G^e)m_{ki}\Big]\mathcal{I}_j(1;G^e)-\phi'(g^e_{ij})=0.$$

Clearly, we have $g^e_{ij}>g_{ij}$ for each $i,j\in N$, $i\neq j$. This occurs because, when an agent forms the links, she ignores the positive externality on the efforts of others. The discrepancy between equilibrium and efficient arrangements in our setting echoes with Jackson and Wolinsky (1996)'s observation that a tension generally persists between stable networks and the efficient one.

Nevertheless, the spontaneous network is again a nested core-periphery network.

Proposition 4 (Spontaneous network)**.** *Consider the strategic network formation game. In every subgame perfect equilibrium* $(\boldsymbol{x},G)$*,* G *is a nested core-periphery network.*

However, note that the set of equilibrium networks contains the special cases of non-hierarchical, complete networks such that all links have identical weight. In these networks, all agents have the same (inward- and outward-) centrality, exert the same amount of effort, and obtain the same payoffs. I leave the question regarding whether hierarchical networks or non-hierarchical ones are more stable equilibrium networks for future research.

6.2 Comparison between nested core-periphery networks and nested split graphs

In this subsection, I compare my characterization with the one obtained by Belhaj et al. (2016) and Hiller (2017). The two papers show that, when restricting to simple networks, the optimal network in games with complementarity is a nested split graph. I first show that nested split graphs are special cases of nested core-periphery networks. Then I show that a distinct feature of nested split graphs need not hold for nested core-periphery networks in general.

The definition of nested split graphs has been given previously. To remind the reader, let d_i denote the degree of i in network G, i.e., $d_i=\sum_j g_{ij}$. A network $G=(g_{ij})$ is called a **nested split graph** if G is simple, and the neighborhoods of nodes are nested such that, for each i, j and $k \neq i, j$, if $d_i \geq d_j$ and $g_{jk} = 1$, then $g_{ik} = 1$. The following observation follows directly from the definitions; thus the proof is omitted.

Proposition 5. *Consider network G. The following are equivalent:*

1. *G is a nested split graph;*

2. *G is a nested core-periphery network that is simple (with unweighted and undirected links).*

To examine more closely the connection between nested split graphs and nested core-periphery networks, I introduce the notion of simple closure of a network, which is a procedure that transforms networks with weighted and directed links to simple networks. A **simple closure** of $G = (g_{ij})$ with cut-off $r > 0$ is a simple network $G^s = (g^s_{ij})$ such that, for each i and j, $g^s_{ij} = 1$ if $\max\{g_{ij}, g_{ji}\} \geq r$, and $g^s_{ij} = 0$ otherwise. Note that, based on this definition, *every* network, except for the empty ones, has more than one simple closures. Therefore, only examining a simple closure of a network necessarily leaves out some information of the original network.

Proposition 6. *Let G be a nested core-periphery network. Every simple closure of G is a nested split graph.*

The above observations highlight the connections between nested split graphs and nested core-periphery networks. The connections suggest that some features—core-periphery division of nodes, nestedness of neighborhoods, and short diameter—are robust features of optimal networks in games with strategic complementarities, regardless of whether the domain is restricted to simple networks.

However, despite the connections, a critical property of nested split graphs does not hold for nested core-periphery networks in general. In a nested split graph, we can use degree centrality d_i to measure the coreness of the node. However, the following example shows that, when expanding the scope to all weighted and directed networks, degree centrality need not induce the same ordering as Bonacich centrality in the optimal network. Indeed, degree centrality may even induce a completely opposite ordering to the one based on $\mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G)$ or $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G)$. When allowing for weighted and directed links, it may occur at optimum that some links are asymmetric; hence, there is not guarantee that different centrality measures induce the same ordering of nodes, and degree-centrality is generally not a correct measure to identify optimal networks.

Example. Consider how to organize the interaction among agents $N = \{1, 2, 3\}$. Let $\alpha_i = i$ for $i \in \{1, 2, 3\}$. To keep the setting as close as possible to Belhaj et al. (2016) and Hiller (2017), let the objective of the planner be $W = \sum x_i$, and the investment in agents $t_i = 0$ for each agent be exogenous. Moreover, let $\phi(g_{ij}) = g_{ij}^2/2$ and $\lambda = 0.1$. Let d_i denote the (in-) degree centrality of i, i.e., $d_i = \sum_j g_{ij}$. Figure 6 shows the optimal network. We have $d_1 > d_2 > d_3$, while $\mathcal{I}_3(\boldsymbol{\alpha}; G) > \mathcal{I}_2(\boldsymbol{\alpha}; G) > \mathcal{I}_1(\boldsymbol{\alpha}; G)$ and $\mathcal{O}_3(1; G) > \mathcal{O}_2(1; G) > \mathcal{O}_1(1; G)$.

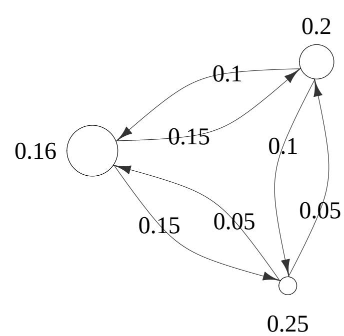


Figure 6: The number on each link shows the weight of the link. The number beside each node shows d_i. The size of each node is proportional to $\mathcal{I}_i(\alpha; G)$. All numbers are rounded to two decimal places.

7 Conclusion

This paper shows that nestedness and hierarchy emerge almost inevitably as consequences of optimizing the structure of an organization, when agents play a game with strategic complementary. Specifically, the optimal interaction structure is characterized by a *nested core-periphery network* with *strict hierarchy.* In a nested core-periphery network, some agents are more central—inwards as well as outwards—than others. The neighborhoods of the agents are nested such that, if an agent with lower centrality links to some agent, then an agent with higher centrality must also link to that agent and the link must be stronger. Hierarchy among agents is strict in the sense that no two agents can have the same centrality. All agents are strictly ranked, leading to unequal payoffs.

Strikingly, even if the agents are homogeneous, and the planner has an arbitrarily concave objective function with respect to the action of each agent, strict hierarchy arises at optimum. The reason for this is that complementarity between the actions of agents generates a fundamental demand for creating a core group of agents to exert a large amount of effort in order to improve even the worst-performing agents. Hence, unless the only objective of the planner is to produce an egalitarian outcome, it never pays off to equalize the performances of any two agents. If hierarchy is the necessary structure of an organization to bring benefits to almost every its member, it is not surprising that it features most real-world organizations, even for those where reducing inequality is an important concern.

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Appendix–Proofs

Throughout the proofs, I simply write

$$\mathcal{O}_i \equiv \mathcal{O}_i(\nabla W(\boldsymbol{x}); G) = \sum_{j \in N} f'(x_j) m_{ji}$$

and

$$\mathcal{I}_i \equiv \mathcal{I}_i(\boldsymbol{\alpha} + v(\boldsymbol{t}); G) = \sum_{j \in N} m_{ij} \left[\alpha_j + v(t_j)\right].$$

The following fact is useful in deriving all subsequent results.

Lemma 2. *Consider the square matrix G whose ijth element is g_{ij}, and let m_{pq} be the pqth element of $M = [I - \lambda G]^{-1}$. Then*

$$\frac{\partial m_{pq}}{\partial g_{ij}} = \lambda m_{pi} m_{jq}.$$

Proof. From standard results of matrix differentiation, we know $\frac{\partial M}{\partial g_{ij}} = -M \frac{\partial [I - \lambda G]}{\partial g_{ij}} M$.

Hence the stated result is the pqth element of

$$\frac{\partial M}{\partial g_{ij}} = - \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & -\lambda & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 0 & \cdots & 0 & m_{1k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & m_{nk} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix}$$

$$= \lambda \begin{bmatrix} m_{1i}m_{j1} & m_{1i}m_{j2} & \cdots & m_{1i}m_{jn} \\ m_{2i}m_{j1} & m_{2i}m_{j2} & \cdots & m_{2i}m_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ m_{ni}m_{j1} & \cdots & \cdots & m_{ni}m_{jn} \end{bmatrix}.$$

□

Proof of Proposition 1

First, I show that he equilibrium effort of i is the inward-centrality of i with base value $\alpha + v(t)$. To start with, obtain the best response function of agent i:

$$x_i = \alpha_i + v(t_i) + \lambda \sum_{j \in N} g_{ij} x_j.$$

Express it in the matrix form:

$$\boldsymbol{x} = \boldsymbol{\alpha} + v(\boldsymbol{t}) + \lambda G \boldsymbol{x}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and $v(\boldsymbol{t}) = (v(t_1), \ldots, v(t_n))$.

Given $\lambda < 1/\mu(G)$, the matrix $[I - \lambda G]$ is invertible, and thus $M = [I - \lambda G]^{-1}$ is well defined. It follows that

$$\boldsymbol{x} = [I - \lambda G]^{-1} [\boldsymbol{\alpha} + v(\boldsymbol{t})] = M [\boldsymbol{\alpha} + v(\boldsymbol{t})],$$

Thus,

$$x_i = \sum_{j \in N} m_{ij} [\alpha_j + v(t_j)] = \mathcal{I}_i(\alpha + v(\boldsymbol{t}); G)$$

by the definition of inward-centrality.

Next, I show that a solution exists to the planner's partial problem, and it has the stated property. First, given the equilibrium effort of the agents, the planner's problem becomes

$$\max_{(t_1,\ldots,t_n)\geq 0} \sum_{i \in N} f\left(\sum_{j \in N} m_{ij} [\alpha_j + v(t_j)]\right)$$

subject to $\sum_i t_i \leq \bar{t}$. Since the objective function is continuous in $t = (t_1, \ldots, t_n)$ and the domain of t is a closed, bounded subset of $\mathbb{R}^n$ (and thus compact), a maximum exists.

To show that agents at positions with greater $\mathcal{O}_i(w'(\boldsymbol{x}); G)$ receive more investment, set up the Lagrangian function

$$\mathcal{L} = \sum_{i \in N} f(x_i) + \theta \left(\bar{t} - \sum_{i \in N} t_i\right),$$

where $x_i = \sum_{j \in N} m_{ij} [\alpha_j + v(t_j)]$ and θ is the Lagrangian multiplier. It follows that, at optimum,

$$\frac{\partial \mathcal{L}}{\partial t_i} = \sum_{j \in N} f'(x_j) m_{ji} v'(t_i) - \theta = \mathcal{O}_i(\nabla W(\boldsymbol{x}); G) v'(t_i) - \theta = 0.$$

Since $v'(0) \to \infty$, at optimum all agents receive positive investment, and thus for each $i \in N$, $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G) v'(t_i) = \theta$. Hence, for each $i, j \in N$,

$$\frac{\mathcal{O}_i(\nabla W(\boldsymbol{x}); G)}{\mathcal{O}_j(\nabla W(\boldsymbol{x}); G)} = \frac{v'(t_j)}{v'(t_i)}.$$

Thus, for each $i, j \in N$,

$$t_i \geq t_j \Leftrightarrow v'(t_j) \geq v'(t_i) \Leftrightarrow \mathcal{O}_i\left(\nabla W(\boldsymbol{x}); G\right) \geq \mathcal{O}_j\left(\nabla W(\boldsymbol{x}); G\right)$$

given $v'' < 0$.

Proof of Lemma 1

Step 1: *If and only if $\mathcal{I}_i(\alpha + v(t); G) \geq \mathcal{I}_j(\alpha + v(t); G)$, then $g_{ki} \geq g_{kj}$ for each $k \neq i, j$. If and only if $\mathcal{O}_i(\nabla W(\boldsymbol{x}); G) \geq \mathcal{O}_j(\nabla W(\boldsymbol{x}); G)$, then $g_{ik} \geq g_{jk}$ for each $k \neq i, j$.*

The Lagrangian function for the planner's problem is

$$\mathcal{L} = \sum_{i \in N} f'(x_i) \sum_{j \in N} m_{ij}\left[\alpha_j + v(t_j)\right] + \sigma\left[\bar{\phi} - \sum_{i,j \in N, i \neq j} \phi(g_{ij})\right] + \theta\left(\bar{t} - \sum_{i \in N} t_i\right),$$

where σ is the Lagrangian multiplier for the constraint on total intensity of interactions. Applying Lemma 2, we obtain

$$\frac{\partial \mathcal{L}}{\partial g_{ij}} = \lambda \mathcal{O}_i \mathcal{I}_j - \sigma \phi'(g_{ij}).$$

Let (t, G) solve the planner's problem. Since $\phi'(0) = 0$, we have $g_{ij} > 0$ and

$$\lambda \mathcal{O}_i \mathcal{I}_j = \sigma \phi'(g_{ij})$$

for each $i, j \in N$, $i \neq j$.

It follows that for each $i, j, p, q \in N$,

$$\begin{aligned} g_{ij} \geq g_{pq} &\Leftrightarrow \sigma \phi'(g_{ij}) \geq \sigma \phi'(g_{pq}) \\ &\Leftrightarrow \lambda \mathcal{O}_i \mathcal{I}_j \geq \lambda \mathcal{O}_p \mathcal{I}_q \end{aligned}$$

given $\phi'' > 0$.

Letting $i = p = k$, we obtain the first stated result:

$$g_{kj} \geq g_{kq} \Leftrightarrow \mathcal{I}_j \geq \mathcal{I}_q.$$

Letting $j = q = k$, we obtain the second stated result:

$$g_{ik} \geq g_{pk} \Leftrightarrow \mathcal{O}_i \geq \mathcal{O}_p.$$

Step 2: *If $\mathcal{O}_i \geq \mathcal{O}_j$, then $\mathcal{I}_i \geq \mathcal{I}_j$.*

Let $\mathcal{O}_i \geq \mathcal{O}_j$. By Step 1, $g_{ik} \geq g_{jk}$ for each $k \in N, k \neq i, j$. Given

$$\mathcal{I}_i = \alpha_i + v(t_i) + \lambda \sum_{k \neq i,j} g_{ik}\mathcal{I}_k + \lambda g_{ij}\mathcal{I}_j$$

we have

$$\begin{aligned}\mathcal{I}_i - \mathcal{I}_j &= \alpha_i + v(t_i) - \alpha_j - v(t_j) + \lambda \sum_{k \neq i,j} (g_{ik} - g_{jk})\mathcal{I}_k + \lambda g_{ij}\mathcal{I}_j - \lambda g_{ji}\mathcal{I}_i \\ &\geq \alpha_i + v(t_i) - \alpha_j - v(t_j) + \lambda g_{ij}\mathcal{I}_j - \lambda g_{ji}\mathcal{I}_i.\end{aligned}$$

By Proposition 3, $\mathcal{O}_i \geq \mathcal{O}_j$ implies $\alpha_i \geq \alpha_j$ and $t_i \geq t_j$.

Now, suppose $\mathcal{I}_i < \mathcal{I}_j$. Then

$$\lambda \left[g_{ji}\mathcal{I}_i - g_{ij}\mathcal{I}_j\right] \geq \alpha_i + v(t_i) - \alpha_j - v(t_j) > 0.$$

It follows that $g_{ji} > g_{ij}$. This contradicts (see the first-order condition for g_{ij} in proof of Lemma 1 for details) that, from $\mathcal{O}_i \geq \mathcal{O}_j$ and $\mathcal{I}_i < \mathcal{I}_j$,

$$\phi'(g_{ij}) = \lambda\mathcal{O}_i\mathcal{I}_j < \lambda\mathcal{O}_j\mathcal{I}_i \leq \phi'(g_{ji}).$$

Hence, we must have $\mathcal{I}_i \geq \mathcal{I}_j$.

Step 3: *If $\mathcal{I}_i \geq \mathcal{I}_j$, then $\mathcal{O}_i \geq \mathcal{O}_j$.*

Instead of showing $\mathcal{I}_i \geq \mathcal{I}_j \Rightarrow \mathcal{O}_i \geq \mathcal{O}_j$ directly, I show that $\mathcal{O}_i < \mathcal{O}_j \Rightarrow \mathcal{I}_i < \mathcal{I}_j$.

Let $\mathcal{O}_i < \mathcal{O}_j$. Observe

$$\mathcal{I}_i = \alpha_i + v(t_i) + \lambda \sum_{k \neq i,j} g_{ik}\mathcal{I}_k + \lambda g_{ij}\mathcal{I}_j.$$

By Step 1, $g_{ik} \leq g_{jk}$ for each $k \neq i, j$. By lemma 4, $\alpha_i \leq \alpha_j$. Moreover, by proposition 1, $t_i < t_j$. Taking together, we have

$$\mathcal{I}_i - \mathcal{I}_j < \lambda g_{ij}\mathcal{I}_j - \lambda g_{ji}\mathcal{I}_i.$$

Now, for contradiction, suppose $\mathcal{I}_i \geq \mathcal{I}_j$. Then $g_{ij}\mathcal{I}_j > g_{ji}\mathcal{I}_i$, which further implies

$$g_{ij} > g_{ji} \geq 0.$$

However, this is not possible because if $\mathcal{I}_i \geq \mathcal{I}_j$ and $\mathcal{O}_i < \mathcal{O}_j$ then $\lambda\mathcal{O}_i\mathcal{I}_j < \lambda\mathcal{O}_j\mathcal{I}_i$. It follows from the Kuhn-Tucker condition for the planner's problem that

$$\phi'(g_{ij}) = \lambda\mathcal{O}_i\mathcal{I}_j < \lambda\mathcal{O}_j\mathcal{I}_i \leq \phi'(g_{ji}),$$

which leads to a contradicting result $g_{ji} < g_{ij}$ given $\phi'' > 0$.

Proof of Proposition 2

Let (G, t) solve the planner's problem. That G is a nested core-periphery network with coreness measure $\mathcal{O}_i$ follows immediately from Lemma 1. The second claim is proved as a lemma as the following.

Lemma 3. *Let (G, t) solve the planner's problem. For each $i, j \in N, i \neq j$, either $\mathcal{I}_i(\alpha + v(t); G) > \mathcal{I}_j(\alpha + v(t); G)$ or $\mathcal{I}_j(\alpha + v(t); G) > \mathcal{I}_i(\alpha + v(t); G)$.*

Proof. Suppose (t, G) solves the planner's problem. Proposition 3 immediately implies that if $\alpha_i \neq \alpha_j$, then $\mathcal{I}_i(\alpha+v(t); G) \neq \mathcal{I}_i(\alpha+v(t); G)$ (notice that proposition 3 is proved without using lemma 3 or proposition 2). Suppose G is such that in which i and j have $\alpha_i = \alpha_j$

and $\mathcal{I}_i(\alpha + v(t); G) = \mathcal{I}_j(\alpha + v(t); G) = \bar{\mathcal{I}}$, and there exists $k \neq i, j$ with $g_{ik} > 0$. Without loss of generality, let $\alpha_i = \alpha_j = 1$. The ability of agent $k \neq i, j$ is denoted by $\alpha_k > 0$. It can be $\alpha_k = 1$ for each k in the network, or each k has a different α_k. For brevity, denote $\mathcal{I}_i = \mathcal{I}_i(\alpha + v(t); G)$ and $\mathcal{O}_i = \mathcal{O}_i\Big(\nabla W(\boldsymbol{x}); G\Big)$. Since $\boldsymbol{t}$ plays no role in the proof, let $t_k = 0$ for each $k \in N$ (an alternative way of thinking about this is that I replace $\alpha_k + t_k$ with α_k). Also, let $x_i^*(G)$ denote the equilibrium effort of i given network G.

By Lemma 1, $\mathcal{I}_i = \mathcal{I}_j$ implies $\mathcal{O}_i = \mathcal{O}_j$, and thus $g_{ij} = g_{ji} = c$ by the first-order conditions for g_{ij} and g_{ji} for some $c > 0$. Moreover, by Lemma 1, we have $g_{ki} = g_{kj} = a > 0$ and $g_{ik} = g_{jk} = b > 0$ for each $k \neq i, j$ for some $a, b > 0$. Now, consider a change of the values of the links g_{ik} and g_{jk} as follows. Let ϵ be such that $0 < \epsilon < \min\{a, b\}$. Let $G^\epsilon = (g_{ij}^\epsilon)$ be the network constructed from $G = (g_{ij})$ such that, for each $k \neq i, j$,

$$\begin{cases} g_{ik}^\epsilon = g_{ik} - \epsilon = b - \epsilon \\ g_{jk}^\epsilon = g_{jk} + \epsilon = b + \epsilon \\ g_{ki}^\epsilon = g_{ki} - \epsilon = a - \epsilon \\ g_{kj}^\epsilon = g_{kj} + \epsilon = a + \epsilon, \end{cases}$$

whereas $g_{k\ell}^\epsilon = g_{k\ell}$ for each $k, \ell \in N$ such that $k \neq i, j$ and $\ell \neq i, j$. I shall show that, for small ϵ, G^ϵ is feasible (it does not exceed the maximum total intensity of interactions), but strictly improves the objective function of the planner over G. Hence, the initial network G is not optimal.

To see that G^ϵ is feasible, observe that, for small $\epsilon > 0$,

$$\begin{aligned} \sum_{k,\ell \in N, k \neq \ell} \phi(g_{k\ell}^\epsilon) - \sum_{k,\ell \in N, k \neq \ell} \phi(g_{k\ell}) &= \epsilon \sum_{k \neq i,j} [\phi'(g_{jk}) - \phi'(g_{ik}) + \phi'(g_{kj}) - \phi'(g_{ki})] \\ &= \epsilon \sum_{k \neq i,j} [\phi'(b) - \phi'(b) + \phi'(c) - \phi'(c)] \\ &= 0. \end{aligned}$$

Hence, if G is feasible, then G^ϵ is also feasible for small $\epsilon > 0$.

Now, I show that G^ϵ strictly improves the objective function of the planner over G. I shall argue that the equilibrium effort under G^ϵ, $x_k^*(G^\epsilon)$, is strictly greater, i.e., $x_k^*(G^\epsilon) > x_k^*(G)$ for each $k \neq i$. The only agent who exerts less effort is i. However, the increase of effort of j fully compensate the reduction of effort of i.

Let $\boldsymbol{x}^\tau$, for $\tau = 0, 1, 2, \ldots$, be a sequence that converges to $x_k^*(G^\epsilon)$ constructed as follows. For $\tau = 0$ and each $k \neq i, j$, let $x_k^0 = x_k^*(G)$. For i and j, let x_i^0 and x_j^0 be defined by

$$\begin{cases} x_i^0 = 1 + \lambda g_{ij}^\epsilon x_j^0 + \lambda g_{ik}^\epsilon \sum_{k \neq i,j} x_k^*(G) \\ x_j^0 = 1 + \lambda g_{ji}^\epsilon x_i^0 + \lambda g_{jk}^\epsilon \sum_{k \neq i,j} x_k^*(G), \end{cases}$$

i.e.,

$$\begin{cases} x_i^0 = 1 + \lambda c x_j^0 + \lambda (a - \epsilon) \sum_{k \neq i,j} x_k^*(G) \\ x_j^0 = 1 + \lambda c x_i^0 + \lambda (a + \epsilon) \sum_{k \neq i,j} x_k^*(G). \end{cases}$$

Notice the equilibrium efforts of i and j under G, $x_i^*(G)$ and $x_j^*(G)$, satisfy

$$\begin{cases} x_i^*(G) = 1 + \lambda c x_j^*(G) + \lambda a \sum_{k \neq i,j} x_k^*(G) \\ x_j^*(G) = 1 + \lambda c x_i^*(G) + \lambda a \sum_{k \neq i,j} x_k^*(G). \end{cases}$$

Hence

$$x_i^0 + x_j^0 = \frac{1}{1 - \lambda c} \left[2 + 2\lambda a \sum_{k \neq i,j} x_k^*(G) \right] = x_i^*(G) + x_j^*(G) = 2\bar{\mathcal{I}},$$

and

$$x_j^0 - x_i^0 = \lambda c (x_i^0 - x_j^0) + 2\lambda\epsilon \sum_{k \neq i,j} x_k^*(G) = \frac{2\lambda\epsilon}{1 + \lambda c} \sum_{k \neq i,j} x_k^*(G) > 0.$$

For $\tau \geq 1$, $\boldsymbol{x}^\tau$ is defined by, for each $\ell \in N$,

$$x_\ell^\tau = \alpha_\ell + \lambda \sum_{k \in N} g_{\ell k}^\epsilon x_k^{\tau - 1}.$$

I show that $\boldsymbol{x}^\tau$ is an increasing sequence by induction. First, we check $\tau = 1$. Clearly, $x_i^1 = x_i^0$ and $x_j^1 = x_j^0$. Hence, $x_i^1 + x_j^1 \geq x_i^*(G) + x_j^*(G)$, and $x_j^1 - x_i^1 \geq x_j^0 - x_i^0 > 0$. For

$k \neq i, j,$

$$\begin{aligned} x_k^1 &= \alpha_k + \lambda b \left[x_i^0 + x_j^0 \right] + \lambda \sum_{\ell \neq i,j} g_{k\ell} x_\ell^0 + \lambda \epsilon \left(x_j^0 - x_i^0 \right) \\ &= \alpha_k + \lambda b \left[x_i^*(G) + x_j^*(G) \right] + \lambda \sum_{\ell \neq i,j} g_{k\ell} x_\ell^*(G) + \lambda \epsilon \left(x_j^0 - x_i^0 \right) \\ &= x_k^0 + \lambda \epsilon \left(x_j^0 - x_i^0 \right). \end{aligned}$$

Hence

$$x_k^1 - x_k^0 = \lambda \epsilon \left(x_j^0 - x_i^0 \right) = \epsilon^2 \left(\frac{2\lambda^2}{1 + \lambda c} \right) \sum_{\ell \neq i,j} x_\ell^*(G) > 0. \tag{8}$$

For $\tau \geq 2$, suppose $x_k^{\tau-1} \geq x_k^{\tau-2}$ for each $k \in N$. Then

$$\begin{aligned} x_i^\tau &= 1 + \lambda c x_j^{\tau-1} + \lambda (a - \epsilon) \sum_{k \neq j} x_k^{\tau-1} \\ &\geq 1 + \lambda c x_j^{\tau-2} + \lambda (a - \epsilon) \sum_{k \neq j} x_k^{\tau-2} \\ &= x_i^{\tau-1} \end{aligned}$$

and

$$\begin{aligned} x_j^\tau &= 1 + \lambda c x_i^{\tau-1} + \lambda (a + \epsilon) \sum_{k \neq j} x_k^{\tau-1} \\ &\geq 1 + \lambda c x_i^{\tau-2} + \lambda (a + \epsilon) \sum_{k \neq j} x_k^{\tau-2} \\ &= x_j^{\tau-1}. \end{aligned}$$

And for each $k \neq i, j$,

$$
\begin{aligned}
x_k^\tau &= \alpha_k + \lambda(b+\epsilon)x_i^{\tau-1} + \lambda(b-\epsilon)x_j^{\tau-1} + \lambda \sum_{\ell \neq i,j} g_{k\ell} x_\ell^{\tau-1} \\
&\geq \alpha_k + \lambda(b+\epsilon)x_i^{\tau-2} + \lambda(b-\epsilon)x_j^{\tau-2} + \lambda \sum_{\ell \neq i,j} g_{k\ell} x_\ell^{\tau-2} \\
&= x_k^{\tau-1}.
\end{aligned}
$$

Now I show $\boldsymbol{x}^\tau$ converges to $\boldsymbol{x}^*(G^\epsilon)$. Let λ^τ be the τth power of λ and $[G^\epsilon]^\tau$ be the τth power of G^ϵ. By assumption 4, a unique $\boldsymbol{x}^*(G^\epsilon)$ exists such that $\boldsymbol{x}^*(G^\epsilon) = [\sum_{t=0}^\infty \lambda^\tau [G^\epsilon]^\tau] \boldsymbol{\alpha}$. Observe from $\boldsymbol{x}^\tau = \boldsymbol{\alpha} + \lambda G^\epsilon \boldsymbol{x}^{\tau-1}$ that

$$
\lim_{t\to\infty} \boldsymbol{x}^\tau = \left[\sum_{t=0}^{\infty} \lambda^\tau [G^\epsilon]^\tau\right] \boldsymbol{\alpha} + \lim_{t\to\infty} \lambda^\tau [G^\epsilon]^\tau x^0.
$$

By assumption 4, $\sum_{t=0}^\infty \lambda^\tau [G^\epsilon]^\tau$ converges. Thus, $\lim_{t\to\infty} \lambda^\tau [G^\epsilon]^\tau = 0$. Therefore,

$$
\lim_{t\to\infty} \boldsymbol{x}^\tau = \left[\sum_{t=0}^{\infty} \lambda^\tau [G^\epsilon]^\tau\right] \boldsymbol{\alpha} = \boldsymbol{x}^*(G^\epsilon).
$$

Since $\boldsymbol{x}^\tau$ is an increasing sequence converging to $\boldsymbol{x}^*(G^\epsilon)$, $x_k^*(G^\epsilon) \geq x_k^1 > x_k^*(G)$ for each $k \neq i, j$. Also, $x_i^*(G^\epsilon) + x_j^*(G^\epsilon) \geq x_i^1 + x_j^1 = x_i^*(G) + x_j^*(G)$. From the Taylor's Theorem, we have

$$
\begin{aligned}
W(\boldsymbol{x}^*(G^\epsilon)) - W(\boldsymbol{x}^*(G)) = f'(x_i(G)) &[x_i^*(G^\epsilon) - x_i(G)] \\
&+ f'(x_j(G)) \left[x_j^*(G^\epsilon) - x_j(G)\right] \\
&+ \sum_{k \neq i,j} f'(x_k^*(G^\epsilon)) [x_k^*(G^\epsilon) - x_k^*(G)] + R(\epsilon)
\end{aligned}
$$

where $R(\epsilon)$ is the remainder term of the Taylor polynomial, which goes to zero faster than the first-order changes as $\epsilon \to 0$. Hence, given $x_i^*(G) = \bar{\mathcal{I}} = x_j^*(G)$ and equation (8), there exists $\epsilon > 0$ such that

$$\begin{aligned} W(\boldsymbol{x}^*(G^\epsilon)) - W(\boldsymbol{x}^*(G)) &= f'(x_i(G)) \left[\left(x_i^*(G^\epsilon) + x_j^*(G^\epsilon) \right) - \left(x_i^*(G) + x_j^*(G) \right) \right] \\ &\qquad + \sum_{k \neq i,j} f'(x_k^*(G^\epsilon)) \left[x_k^*(G^\epsilon) - x_k^*(G) \right] + R(\epsilon) \\ &\geq \sum_{k \neq i,j} f'(x_k^*(G^\epsilon)) \left[x_k^*(G^\epsilon) - x_k^*(G) \right] + R(\epsilon) \\ &> 0. \end{aligned}$$

□

Proof of Proposition 3

Lemma 4 below shows that $\alpha_i > \alpha_j$ implies $\mathcal{O}_i \geq \mathcal{O}_j$. Lemma 5 shows that $\alpha_i > \alpha_j$ combined with $\mathcal{O}_i \geq \mathcal{O}_j$ implies $\mathcal{I}_i > \mathcal{I}_j$. By lemma 1, $\mathcal{I}_i > \mathcal{I}_j$ if and only if $\mathcal{O}_i > \mathcal{O}_j$. Hence, $\alpha_i > \alpha_j$ implies both $\mathcal{I}_i > \mathcal{I}_j$ and $\mathcal{O}_i > \mathcal{O}_j$. Note that this result is proved without using lemma 3 or proposition 2.

Lemma 4. *Let (t, G) solve the planner's problem. If $\alpha_i > \alpha_j$, then $\mathcal{O}_i \geq \mathcal{O}_j$.*

Proof. For brevity, denote $\mathcal{O}_i \equiv \mathcal{O}_i(w'(\boldsymbol{x}); G)$ and $\mathcal{I}_i \equiv \mathcal{I}_i(\alpha + v(t); G)$. Let $\alpha_i > \alpha_j$. Suppose $\mathcal{O}_i < \mathcal{O}_j$. Given each agent exerting their equilibrium effort, the planner's objective function is

$$W = \sum_{i \in N} f \left(\sum_{j \in N} m_{ij} \left[\alpha_j + v(t_j) \right] \right).$$

and thus

$$W_{\alpha_j} = \sum_{i \in N} f'(x_i) m_{ij} = \mathcal{O}_j(\nabla W(\boldsymbol{x}); G)$$

Swapping agents i and j (while keeping the shape of the networks and the investment the same to their original positions) results in the change in W

$$dW = (\alpha_j - \alpha_i)\mathcal{O}_i + (\alpha_i - \alpha_j)\mathcal{O}_j = (\alpha_i - \alpha_j)(\mathcal{O}_j - \mathcal{O}_i) > 0.$$

Thus, the original allocation of agents is not optimal. $\square$

Lemma 5. *Suppose* $(G, \boldsymbol{t})$ *solves the planner's problem. If* $\alpha_i > \alpha_j$ *and* $\mathcal{O}_i \geq \mathcal{O}_j$*, then* $\mathcal{I}_i > \mathcal{I}_j$.

Proof. Let $\mathcal{O}_i \geq \mathcal{O}_j$. By lemma 1, $g_{ik} \geq g_{jk}$ for each $k \neq i, j$. By proposition 1, $t_i \geq t_j$. From

$$\mathcal{I}_i = \alpha_i + v(t_i) + \lambda \sum_{k \neq i,j} g_{ik}\mathcal{I}_k + \lambda g_{ij}\mathcal{I}_j$$

we obtain

$$\mathcal{I}_i - \mathcal{I}_j \geq \alpha_i - \alpha_j + \lambda g_{ij}\mathcal{I}_j - \lambda g_{ji}\mathcal{I}_i.$$

Given $\alpha_i > \alpha_j$, it follows that $\mathcal{I}_i - \mathcal{I}_j > \lambda g_{ij}\mathcal{I}_j - \lambda g_{ji}\mathcal{I}_i$.

Now, for contradiction, suppose $\mathcal{I}_i \leq \mathcal{I}_j$. Then $g_{ij}\mathcal{I}_j < g_{ji}\mathcal{I}_i$, which further implies

$$0 \leq g_{ij} < g_{ji}.$$

However, this is not possible because if $\mathcal{I}_i \leq \mathcal{I}_j$ and $\mathcal{O}_i \geq \mathcal{O}_j$ then $\lambda\mathcal{O}_i\mathcal{I}_j \geq \lambda\mathcal{O}_j\mathcal{I}_i$. It follows from the Kuhn-Tucker condition for the planner's problem (see the proof of Lemma 1 for details) that

$$\phi'(g_{ij}) = \lambda\mathcal{O}_i\mathcal{I}_j \geq \lambda\mathcal{O}_j\mathcal{I}_i = \phi'(g_{ji}),$$

which leads to a contradicting result $g_{ji} \geq g_{ij}$ given $\phi'' > 0$. $\square$

Proof of Proposition 4

Denote $\mathcal{I}_i = \mathcal{I}_i(1; G)$. Suppose i and j are such that $\mathcal{I}_i \geq \mathcal{I}_j$, and consider anther agent $k \neq i, j$. From the first-order condition (7) for g_{ki} and g_{kj} we obtain

$$\phi'(g_{ki}) = \lambda\mathcal{I}_k m_{kk}\mathcal{I}_i \geq \lambda\mathcal{I}_k m_{kk}\mathcal{I}_i = \phi'(g_{kj}).$$

Since $\phi'' > 0$, if and only if $\phi'(g_{ki}) \geq \phi'(g_{kj})$ then $g_{ki} \geq g_{kj}$. Hence, $g_{ki} \geq g_{kj}$.

Next, suppose $\mathcal{I}_i m_{ii} < \mathcal{I}_j m_{jj}$. Then

$$\phi'(g_{ik}) = \lambda \mathcal{I}_i m_{ii} \mathcal{I}_k < \lambda \mathcal{I}_j m_{jj} \mathcal{I}_k = \phi'(g_{jk})$$

and thus $g_{ik} < g_{jk}$ for each $k \neq i, j$. It follows from $\mathcal{I}_i \geq \mathcal{I}_j$ that

$$1 + \lambda g_{ij} \mathcal{I}_j + \lambda \sum_{k \neq i,j} g_{ik} \mathcal{I}_k \geq 1 + \lambda g_{ji} \mathcal{I}_i + \lambda \sum_{k \neq i,j} g_{jk} \mathcal{I}_k.$$

Thus,

$$g_{ij} \mathcal{I}_j - g_{ji} \mathcal{I}_i \geq \sum_{k \neq i,j} (g_{jk} - g_{ik}) \mathcal{I}_k > 0.$$

Hence, $g_{ij} > g_{ji}$. But this is a contradiction with

$$\phi'(g_{ij}) = \lambda \mathcal{I}_{ii} m_{ii} \mathcal{I}_j < \lambda \mathcal{I}_{jj} m_{jj} \mathcal{I}_i = \phi'(g_{ji}).$$

To conclude, $\mathcal{I}_i \geq \mathcal{I}_j$ implies $\mathcal{I}_i m_{ii} \geq \mathcal{I}_j m_{jj}$, and thus we also have $g_{ik} \geq g_{jk}$ for each $k \neq i, j$, since

$$\phi'(g_{ik}) = \lambda \mathcal{I}_i m_{ii} \mathcal{I}_k \geq \lambda \mathcal{I}_j m_{jj} \mathcal{I}_k = \phi'(g_{jk}).$$

Proof of Proposition 6

Let G be a nested core-periphery network with coreness measure c. Let $G^s = (g^s_{ij})$ be a simple closure of G with cut-off r. By definition, G^s is a simple network. Since a nested split graph is a simple nested core-periphery network, I only need to show that G^s is a nested core-periphery network.

Let $d_i(G^s) = \sum_k g^s_{ik}$ denote the degree of i. Consider two nodes i and j, and another node $k \neq i, j$. Suppose $d_i(G^s) \geq d_j(G^s)$ and $g^s_{jk} = 1$. I show that we must also have $g^s_{ik} = 1$. It follows that G^s is a nested core-periphery network with $d_i(G^s)$ as the coreness measure.

Suppose $g^s_{ik} = 0$. Then $\max\{g_{ik}, g_{ki}\} < r < \max\{g_{jk}, g_{kj}\}$. Then we must have $c_i < c_j$, since

$$c_i \geq c_j \Rightarrow [g_{ik} \geq g_{jk} \text{ and } g_{ki} \geq g_{kj}] \Rightarrow \max\{g_{ik}, g_{ki}\} \geq \max\{g_{jk}, g_{kj}\}$$

given G is a nested core-periphery network. It follows that, for each $\ell \neq i, j$, we have $g_{i\ell} \leq g_{j\ell}$ and $g_{\ell i} \leq g_{\ell j}$. Thus, $\max\{g_{i\ell}, g_{\ell i}\} \leq \max\{g_{j\ell}, g_{\ell j}\}$ and thereby $g^s_{i\ell} \leq g^s_{i\ell}$ for each $\ell \neq i, j$. Also note that $g^s_{ij} \leq g^s_{ji}$. A contradiction then follows:

$$d_i(G^s) = \sum_{\ell \neq k} g^s_{i\ell} + g^s_{ik} < \sum_{\ell \neq k} g^s_{j\ell} + g^s_{jk} = d_j(G^s).$$